Maximal chains in second-order arithmetic

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Height

Let P be a well-founded partial order. By recursion we define a function $\mathrm{ht}\colon P \to \mathrm{On}$ by letting for all $y \in P$

$$\mathrm{ht}(y) = \sup\{\mathrm{ht}(x) + 1 \colon x <_P y\} = \{\mathrm{ht}(x) \colon x <_P y\}.$$

The **height** of P is

$$\mathrm{ht}(P) = \sup\{\mathrm{ht}(x) + 1 \colon x \in P\} = \{\mathrm{ht}(x) \colon x \in P\}.$$

A chain C on P is **maximal** if its order type equals $\mathrm{ht}(P)$.

A chain C on P is **strongly maximal** if for every $\beta < \mathrm{ht}(P)$ there exists $x \in C$ such that $\mathrm{ht}(x) = \beta$.

Theorems

Theorem (Wolk 1967)

Every wpo has a strongly maximal chain.

Theorem

Let P be a well-founded partial order such that

$$P_\beta = \{x \in P \colon \operatorname{ht}(x) = \beta\}$$

is finite for all $\beta < \operatorname{ht}(P)$. Then P has a strongly maximal chain.

Theorem (Schmidt 1981)

Let P be a well-founded partial such that $\{\operatorname{ht}(x) \colon x \in A\}$ is finite for every antichain A of P. Then there exists a maximal chain.

Formalization

The height of a well-founded partial order is not definable in RCA_0. Actually, the existence of the height is equivalent to ATR_0 over RCA_0. The following version of Wolk's theorem has been studied by Marcone and Shore:

> "Every wpo has a chain such that every other chain embeds into it"

No mention of the height. Existence of a maximal chain. The above statement is equivalent to ATR_0 as well.

However, if we want to study these theorems, we must (can) talk about the height. A possible approach is to consider partial orders along with their height. Hence, the notion of **ranked partial order**.

Definition (RCA_0)

A **ranked partial order** is a triple $\mathcal{P} = (P, h, \alpha)$, where P is a partial order, α is a **linear order** and $h \colon P \to \alpha$ is a function from P **onto** α which satisfies

$$h(y) = \sup\{h(x) + 1 \colon x <_P y\} \text{ for all } y \in P,$$

that is:

(H1) $x <_P y$ implies $h(x) < h(y)$

(H2) $\beta < h(y)$ implies $\beta \leq h(x)$ for some $x <_P y$.

We say that $\mathcal{P}$ has **height** α.

We could require h to satisfy the strongest condition

$$h(y) = \{h(x) \colon x <_P y\} \text{ for all } y \in P,$$

that is:

(H1) $x <_P y$ implies $h(x) < h(y)$

(H3) $\beta < h(y)$ implies $\beta = h(x)$ for some $x <_P y$.

Lemma (RCA_0)

Let $\mathcal{P} = (P, h, \alpha)$ be a ranked partial order. Then P is well-founded iff α is well-ordered iff h satisfies (H3).

Definition (RCA_0)

Let $\mathcal{P} = (P, h, \alpha)$ be a ranked partial order. We say that a chain $C \subseteq P$ is

- **strongly maximal** if for all $\beta < \alpha$ there exists a (necessarily unique) $x \in C$ with $h(x) = \beta$;
- **maximal** if there exists an embedding from α to C.

Statements

SMC$^+$ Let $\mathcal{P} = (P, h, \alpha)$ be a ranked partial order. If P is well-founded and $P_\beta = \{x \in P \colon h(x) = \beta\}$ is finite for every $\beta < \alpha$, then $\mathcal{P}$ has a **strongly maximal chain**.

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SMC Let $\mathcal{P} = (P, h, \alpha)$ be a ranked partial order. If P is a wpo, then $\mathcal{P}$ has a **strongly maximal chain**.

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Statements

- **SMC$^+$** Let $\mathcal{P} = (P, h, \alpha)$ be a ranked partial order. If P is well-founded and $P_\beta = \{x \in P \colon h(x) = \beta\}$ is finite for every $\beta < \alpha$, then $\mathcal{P}$ has a **strongly maximal chain**.
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- **SMC** Let $\mathcal{P} = (P, h, \alpha)$ be a ranked partial order. If P is a wpo, then $\mathcal{P}$ has a **strongly maximal chain**.
- **MC** Let $\mathcal{P} = (P, h, \alpha)$ be a ranked partial order. If P is a wpo, then $\mathcal{P}$ has a **maximal chain**.
- **MCA** Let $\mathcal{P} = (P, h, \alpha)$ be a ranked partial order. If P is well-founded and $\{h(x) \colon x \in A\}$ is finite for every antichain A of P, then $\mathcal{P}$ has a **maximal chain**.

Trivially,

- $\mathsf{SMC}^+ \to \mathsf{SMC} + \mathsf{MC}^+$
- $\mathsf{SMC} \to \mathsf{MC}$
- $\mathsf{MC}^+ \to \mathsf{MC}$
- $\mathsf{MCA} \to \mathsf{MC}$

Theorem (RCA_0)

The following are equivalent:

1. ACA_0
2. SMC^+
3. SMC
4. MC^+

Question

Does MC *imply* ACA_0*?*

Question

What is the strength of MCA*?*

The classical proof of MCA "works" in ATR_0. The original inductive argument requires some technical annoying adjustments (I still have to write the details).

We skip the proof of SMC^+ in ACA_0. The classical proof is based on Rado's selection lemma, which is equivalent to ACA_0.

SMC $\rightarrow$ ACA_0.

Assume

SMC Let $\mathcal{P} = (P, h, \alpha)$ be a ranked partial order. If P is a wpo, then $\mathcal{P}$ has a strongly maximal chain.

We build a partial order P of height ω^2. P is an ω-sum of partial orders P_m's, each of height ω. The role of P_m is to code whether $m \in \operatorname{ran} f$. If $m \notin \operatorname{ran} f$, P_m is the disjoint sum of an ω-chain $a_m <_P a_{m,0} <_P a_{m,1} <_P \ldots$ and a single point b_m. If $f(n) = m$, P_m is the disjoint sum of a finite chain $a_m <_P a_{m,0} <_P < a_{m,n-1}$ and an ω-chain $b_m <_P b_{m,n} <_P b_{m,n+1} <_P b_{m,n+2} \ldots$ It's clear how to define h.

Given a strongly maximal chain C, we have $m \in \operatorname{ran} f$ iff $b_m \in C$. □

$\mathsf{MC}^+ \to \mathsf{ACA}_0$.

Assume

MC$^+$ Let $\mathcal{P} = (P, h, \alpha)$ be a ranked partial order. If P is well-founded, and $P_\beta = \{x \in P \colon h(x) = \beta\}$ is finite for every $\beta < \alpha$, then $\mathcal{P}$ has a maximal chain.

Let $P = (T, \subseteq)$, where $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is defined by $\sigma \in T$ if and only if for all $m < |\sigma|$

- $\sigma(m) = 0 \to (\forall n < |\sigma|) f(n) \neq m$ (m hasn't entered ran f yet)
- $\sigma(m) = n + 1 > 0 \to f(n) = m$ (σ witnesses $m \in$ ran f)

It's not difficult to show that $\mathcal{P} = (P, h, \omega)$, where $h(\sigma) = |\sigma|$, is a ranked partial order (of height ω) with finite levels.
A maxinal chain is an ω-chain, from which we can compute (the unique) path of T, and hence the range of f. □

It turns out that:

- $\mathsf{SMC}(\omega^2) \to \mathsf{ACA}_0$
- $\mathsf{MC}^+(\omega) \to \mathsf{ACA}_0$.

Question

What about $\mathsf{SMC}(\omega)$, $\mathsf{MC}(\omega)$, *and* $\mathsf{MCA}(\omega)$*?*

Remark

We can define the height of a partial order P by **primitive recursion**, *whenever $x <_P y$ implies $x < y$. Define $h\colon P \to \omega$ by*

$$h(y) = \sup\{h(x) + 1 \colon x <_P y\}.$$

By definition, (H1) and (H2) hold. The range of h is an initial segment of ω.

Lemma

Over RCA_0*, the following are equivalent:*

- CAC
- $\mathsf{MCA}(\omega)$
- $\mathsf{MC}(\omega)$

We only show the reversal.

$\mathsf{MC}(\omega) \rightarrow \mathsf{CAC}$

Bootstrap: $\mathsf{MC}(\omega) \rightarrow \mathsf{B}\boldsymbol{\Sigma}^0_2$. We omit the proof.

Let us prove st-RT^2_2. Let $f : [\mathbb{N}]^2 \rightarrow 2$ be transitive on 0, that is $(\forall x < y < z)(f(x, y) = f(y, z) = 0 \rightarrow f(x, z) = 0)$.

Define a partial order P by

$$x <_P y \text{ if } x < y \wedge f(x, y) = 0.$$

Define $h\colon P \to \omega$ by $h(y) = \sup\{h(x) + 1 \colon x <_P y\}$.

- h is bounded. By $\mathrm{B}\mathbf{\Sigma}^0_2$ there exists $n < \omega$ such that $H = \{x \in P \colon h(x) = n\}$ is infinite. Then H is an infinite antichain on P, and so an infinite 1-homogeneous set for f.
- P is not a wpo. Let (x_n) be a bad sequence. We may assume $n < m$ implies $x_n < x_m$. Then $\{x_n \colon n \in \mathbb{N}\}$ is an infinite 1-homogeneous set for f.
- $\mathcal{P} = (P, h, \omega)$ is a ranked partial order of height ω and P is a wpo. By $\mathrm{MC}(\omega)$, there exists an ω-chain, say (x_n) such that $n < m$ implies $x_n <_P x_m$. Then $\{x_n \colon n \in \mathbb{N}\}$ is an infinite 0-homogeneous set for f.

We don't know whether SMC(ω) already implies ACA_0. The feeling is that SMC(ω) is stronger than MC(ω) (which is equivalent to CAC). We show that SMC(ω) implies the following weak version of Ramsey theorem (in Dorais' blog this is called "Mixed Ramsey theorem"):

> For every coloring $f\colon [\mathbb{N}]^2 \to 2$ there exists either an infinite 0-homogeneous path (that is an infinite set $\{x_0 < x_1 < \dots\}$ such that $f(x_n, x_{n+1}) = 1$ for all n), or an infinite 1-homogeneous set

This statement easily implies CAC. Its strength seems to be unknown.

Proof

Given $f\colon [\mathbb{N}]^2 \to 2$, define $x <_P y$ if $x < y$ and there exists $x = a_0 < a_1 < \ldots < a_k = y$ such that $f(a_i, a_{i+1}) = 0$ for all i. By recursion define $h\colon \mathbb{N} \to \mathbb{N}$ such that $h(y) = \sup\{h(x)\colon x <_P y\}$.

- Suppose h is bounded. By $\mathrm{B}\boldsymbol{\Sigma}^0_2$, P has an infinite antichain, which is an infinite 1-homogeneous set for f.
- Suppose P is not a wpo, and let (x_k) be a bad sequence. We may assume that (x_k) is an increasing sequence of natural numbers. Then $\{x_k\colon k \in \mathbb{N}\}$ is an infinite 1-homogeneous set for f.

- Suppose $\mathcal{P} = (P, h, \omega)$ is a ranked partial order of height ω and P is a wpo. By SMC(ω), there exists a strongly maximal chain, say (x_n) such that $n < m$ implies $x_n <_P x_m$ and $h(x_n) = n$. We claim that $f(x_n, x_{n+1}) = 0$ for all n. Suppose $f(x_n, x_{n+1}) = 1$. As $x_n <_P x_{n+1}$, there exists $x_n = a_0 < a_1 < \ldots < a_k = x_{n+1}$ with $k > 1$ such that $a_i <_P a_{i+1}$ for all i. It follows that $h(x_{n+1}) \geq n + k > n + 1$, a contradiction.

Summary

- $\mathsf{ATR}_0 \to \mathsf{MCA}$
- $\mathsf{ACA}_0 \leftrightarrow \mathsf{SMC}^+ \leftrightarrow \mathsf{SMC} \leftrightarrow \mathsf{SMC}(\omega^2) \leftrightarrow MC^+ \leftrightarrow \mathsf{MC}^+(\omega)$
- $\mathsf{CAC} \leftrightarrow \mathsf{MCA}(\omega) \leftrightarrow \mathsf{MC}(\omega)$

Question

- $\mathsf{SMC}(\omega) \to \mathsf{ACA}_0$?
- $\mathsf{MC} \to \mathsf{ACA}_0$?

Final Remark

Marcone, Montalban, and Shore proved that every hyperarithmetically generic set computes a maximal chain in every computable wpo.
In particular, in every computable wpo there is a maximal chain that does not compute $0'$.
However, it can be still the case that $\mathsf{MC} \to \mathsf{ACA}_0$ by means of a **pseudo** wpo.

Thanks for your attention