Some reverse mathematics results about partial orders

Emanuele Frittaion
University of Udine

May 3, 2012
Notre Dame Logic Seminar

Finite Antichain Condition

A partial order is FAC (Finite Antichain Condition) if it has no infinite antichains.

A WPO (Well Partial Order) is a well founded (no infinite descending sequences) FAC partial order.

For example, linear orders are FAC and well orders are well partial orders.

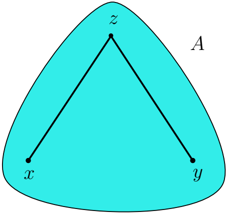
Some WPOs (up to equivalence):

- Finite strings over a finite alphabet (Higman 1952).
- Finite trees (Kruskal 1960).
- Countable linear orders (Laver 1971)
- Finite graphs (Robertson and Seymour 2004)

A characterization of FAC

An initial interval I of P is a downward closed subset of P.

An ideal A of P is an initial interval such that for any $x, y \in A$ there exists $z \in A$ with $x, y \leq_P z$.



A characterization of FAC

Initial interval = downward closed subset.

Ideal = initial interval such that any two elements have a common upper bound.

Theorem (Erdös-Tarski 1943, Bonnet 1973)

A partial order is FAC iff every initial interval is a finite union of ideals.

You can state this in terms of final intervals and filters.

Reverse Mathematics of ETB

Consider the two directions:

$\mathsf{ETB}^{\rightarrow}$ *If P is FAC, then every initial interval of P is a finite union of ideals.*

$\mathsf{ETB}^{\leftarrow}$ *If P is not FAC, then there exists an initial interval which is not a finite union of ideals.*

Theorem (Marcone and F)

The following hold:

- *Over* RCA_0, $\mathsf{ETB}^{\rightarrow}$ *is equivalent to* ACA_0
- $\mathsf{ETB}^{\leftarrow}$ *is provable in* WKL_0*;*
- $\mathsf{ETB}^{\leftarrow}$ *is not provable in* RCA_0 *(hint by Igusa G.).*

Reverse Mathematics of $\mathsf{ETB}^{\rightarrow}$

Theorem

The following are pairwise equivalent over RCA_0.

1. ACA_0;
2. *Every WPO is a finite union of ideals;*
3. $\mathsf{ETB}^{\rightarrow}$: *Every initial interval of a FAC partial order is a finite union of ideals.*

The original proof uses (some form of) choice. The proof in ACA_0 requires a new argument.

Notice that $\mathsf{ETB}^{\rightarrow}$ clearly implies 2.

Reverse mathematics of $\mathrm{ETB}^{\rightarrow}$

Proof.

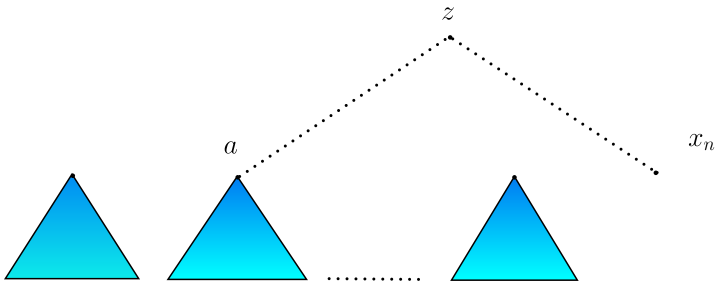
We show that ACA_0 proves that every WPO is a finite union of ideals.

- Let $P = \{x_n \colon n \in \mathbb{N}\}$ be an infinite WPO.
- By recursion, for every $n \in \mathbb{N}$, we define k_n principal ideals $A^n_0, A^n_1, \ldots$.
- A principal ideal is $(a) = \{x \in P \colon x \leq_P a\}$.

Reverse mathematics of ETB$^{\rightarrow}$

Proof.

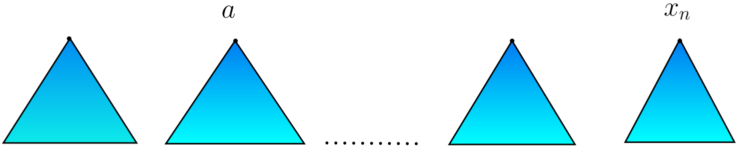
- Stage $n+1$: suppose you have defined k_n principle ideals $A^n_0, A^n_1, \ldots$ and consider x_n.
- If there are $A^n_i = (a)$ and $z \in P$ such that $a, x_n \leq_P z$, let $A^{n+1}_i = (z)$.



Reverse mathematics of ETB$^{\rightarrow}$

Proof.

- Stage $n+1$: suppose you have defined k_n principle ideals $A_0^n, A_1^n, \ldots$ and consider x_n.
- Otherwise, add the principle ideal (x_n).



Reverse mathematics of $\mathrm{ETB}^{\rightarrow}$

Proof.

- Stage $n+1$: suppose you have defined k_n principle ideals $A_0^n, A_1^n, \ldots$ and consider x_n.
- Let $A_i = \bigcup_{n \in \mathbb{N}} A_i^n$.
- Then $P = \bigcup_i A_i$ is a union of ideals.
- Prove that this union is finite: here you use that P is WPO. □

Reverse Mathematics of $\mathrm{ETB}^{\leftarrow}$

Theorem

The statement:

- $\mathrm{ETB}^{\leftarrow}$: *If P is not FAC, then there is an initial interval which is not a finite union of ideals*

is false in the model of computable sets.

(Notice that RCA_0 proves $\mathrm{ETB}^{\leftarrow}$ under the assumption that there is a maximal antichain.)

Reverse Mathematics of $\mathsf{ETB}^{\leftarrow}$

Proof (finite injury).

- We define a partial order P by stages meeting for all e:

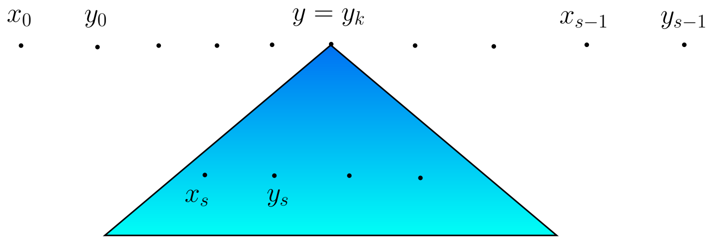
$$R_e\colon (\exists y)\ [(\varphi_e(y) = 1 \to (\forall^\infty z)(z \leq_P y)) \wedge \\ (\varphi_e(y) = 0 \to (\forall^\infty z)(y \leq_P z))]$$

- At stage s we define $\leq_P$ on $P_s = \{x_0, y_0, \ldots, x_{s-1}, y_{s-1}\}$.
- The x_n's yield an infinite computable antichain.
- A single requirement R_e is satisfied by fixing a witness y and waiting for a stage $s+1$ such that $\varphi_{e,s}(y) \in \{0, 1\}$.

Reverse Mathematics of $\mathsf{ETB}^{\leftarrow}$

Proof.

- Stage $s+1$: if $\varphi_{e,s}(y) = 1$, we let $x_s, y_s \leq_P y_k$.



- Transitivity requires attention. □

Reverse Mathematics of $\mathsf{ETB}^{\leftarrow}$

Question

What is the reverse mathematics strength of:

- $\mathsf{ETB}^{\leftarrow}$: If P is not FAC, then there is an initial interval which is not a finite union of ideals.

WKL_0 proves more: in fact the initial interval (a path of a suitable tree) contains the given infinite antichain.

References

Robert Bonnet, *On the cardinality of the set of initial intervals of a partially ordered set*. Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I, North-Holland, Amsterdam, 1975.

P. Erdös and A. Tarski, *On families of mutually exclusive sets*, Ann. of Math. (2), 44:315–329, 1943.

R. Fraïssé. *Theory of relations*, volume 145 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, revised edition, 2000.

Stephen G. Simpson. *Subsystems of second order arithmetic*, *Perspectives in Logic*, Cambridge University Press, Cambridge, 2009.

Scattered linear orders

A linear order is scattered if it does not embed the rationals.

For instance, $\mathbb{Z}$ is scattered, whereas $\mathbb{Z} \cdot \mathbb{Q}$ is not scattered.

It is easy to see that a linear order is scattered iff it has no dense subsets. This can be proved in RCA_0.

Scattered linear orders

A rather less obvious characterization is the following (see Fraïssé):

Theorem

A countable linear order L is scattered iff there exists a countable ordinal α which does not embed into L.

One direction of the theorem is provable in RCA_0, since RCA_0 proves that $\mathbb{Q}$ embeds any countable linear order.

Scattered linear orders

Theorem

A countable scattered linear order does not embed every countable ordinal.

This is not true in general (ω_1 is scattered but embeds every countable ordinal).

Classical proof.

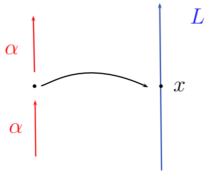
For $x \in L$, let $x^{\downarrow} = \{y \in L \colon y <_L x\}$ and $x^{\uparrow} = \{y \in L \colon y >_L x\}$.

- Let L be a countable linear order. Prove the contrapositive.
- Enumerate $\mathbb{Q} = \{q_0, q_1, \ldots\}$.

Scattered linear orders

Classical proof.

- For every $\alpha < \omega_1$, $\alpha + 1 + \alpha$ embeds into L, and so there is $x \in L$ such that α embeds both into $x^{\downarrow}$ and $x^{\uparrow}$.



- L is countable. Hence there is $x_0 \in L$ such that every $\alpha < \omega_1$ embeds both into $x_0^{\downarrow}$ and into $x_0^{\uparrow}$.
- Keep on embedding $\mathbb{Q}$. □

Reverse Mathematics

Theorem (F and Marcone)

ATR_0 *proves that a countable scattered linear order does not embed every countable linear order.*

The proof uses:

1. Every scattered linear order is embeddable in some $\mathbb{Z}^\alpha$;
2. $\omega^\alpha + 1$ is not embeddable in $\mathbb{Z}^\alpha$.

P. Clote 1989 proved 1 in ATR_0. We prove 2 in ACA_0.

Reverse Mathematics

Questions

What is the reverse mathematics strength of:

- $\omega^\alpha + 1$ is not embeddable in $\mathbb{Z}^\alpha$;
- Every scattered linear order is embeddable in some $\mathbb{Z}^\alpha$;
- A countable scattered linear order does not embed every countable ordinal.

References

P. Clote, *The metamathematics of scattered linear orderings*, volume 29 of *Arch. Math. Logic*, 29(1):9–20, 1989.

R. Fraïssé. *Theory of relations*, volume 145 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, revised edition, 2000.

Stephen G. Simpson. *Subsystems of second order arithmetic*, *Perspectives in Logic*, Cambridge University Press, Cambridge, 2009.

Spilzrajn's Theorem

Szpilrajn's Theorem (1930) says that any partial order has a linear extension. This is a starting point for many natural questions. In most cases, these questions have the following pattern: let $\mathcal{P}$ be some property about partial orders. Then:

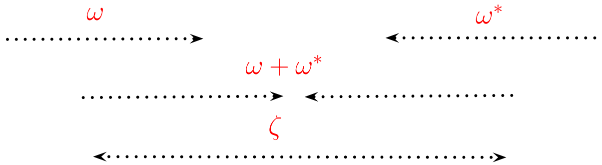
Question

Is it true that any partial order satyisfing $\mathcal{P}$ has a linear extension which also satisfies $\mathcal{P}$?

For example, the *extendibility* problem has this form. A linear order type τ is extendible if every partial order which does not embed τ has a linear extension which does not embed τ either.

For instance, ω, ζ (the integers) and η (the rationals) are extendible.

Linearization of ω, ω^*, $\omega + \omega^*$ and ζ



We say that a countable partial order P is:

- ω-like if every element of P has finitely many predecessors;
- ω^*-like if every element of P has finitely many successors;
- $\omega + \omega^*$-like if every element of P has finitely many predecessors or finitely many successors;
- ζ-like if for every pair of elements $x, y \in P$ there exist only finitely many elements z with $x <_P z <_P y$.

Linearization of ω, ω^*, $\omega + \omega^*$ and ζ

For any $\tau \in \{\omega, \omega^*, \omega + \omega^*, \zeta\}$, we say that τ is linearizable if every τ-like partial order has a τ-like linear extension.

Theorem

The following hold:

1. ω *is linearizable (Milner and Pouzet, see Fraïssé);*
2. ω^* *is linearizable;*
3. $\omega + \omega^*$ *is linearizable;*
4. ζ *is linearizable.*

4. appears to be a new result.

The statements

For $\tau \in \{\omega, \omega^*, \omega + \omega^*, \zeta\}$, we study the statements:

- τ is linearizable: every τ-like partial order has a τ-like linear extension;
- τ is embeddable: every τ-like partial order is embeddable into τ.

The relevant systems are RCA_0, $\mathsf{B}\mathbf{\Sigma}^0_2$ ($\mathbf{\Sigma}^0_2$ bounding principle) and ACA_0.

Results

Theorem (F and Marcone)

Over RCA_0,

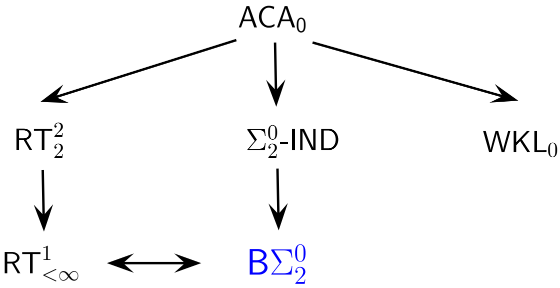
- *"τ is linearizable" is equivalent to:*
  - $\mathsf{B}\boldsymbol{\Sigma}^0_2$ *when* $\tau \in \{\omega, \omega^*, \zeta\}$*;*
  - ACA_0 *when* $\tau = \omega + \omega^*$*.*
- *"τ is embeddable" is equivalent to* ACA_0 *for each* τ*.*

$\mathbf{\Sigma}^0_2$ bounding principle

Recall that $\mathrm{B}\mathbf{\Sigma}^0_2$ is the schema:

$\mathrm{B}\mathbf{\Sigma}^0_2$ $(\forall i < n)(\exists m)\varphi(i, n, m) \Longrightarrow (\exists k)(\forall i < n)(\exists m < k)\varphi(i, n, m)$,

where φ is any $\mathbf{\Sigma}^0_2$ formula.



$\boldsymbol{\Sigma}^0_2$ bounding principle

When proving our equivalences with $\mathsf{B}\boldsymbol{\Sigma}^0_2$, we make use of another equivalent:

FUF $(\forall i < n)(X_i \text{ is finite}) \implies \bigcup_{i<n} X_i$ is finite.

The linearizability of ω is a “genuine” mathematical theorem which turns out to be equivalent to $\mathsf{B}\boldsymbol{\Sigma}^0_2$.

Linearizability of ω

Theorem

"ω is linearizable" is equivalent to $\mathrm{B}\mathbf{\Sigma}^0_2$ *over* RCA_0.

Proof.

Reason in RCA_0 and assume $\mathrm{B}\mathbf{\Sigma}^0_2$ (actually FUF).

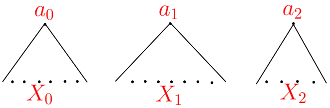
- Let P be an infinite ω-like partial order. By recursion define $(z_n)_n \subseteq P$ such that $z_n \not\leq_P z_i$ for all n and for all $i < n$. Here use FUF and the fact that P is infinite to find z_n.
- Let L be a linear extension of $\sum_{n \in \omega} P_n$, where $P_n = \{x \in P \colon x \leq_P z_n \wedge (\forall i < n)(x \not\leq_P z_i)\}$. Prove that L is an ω-like linear extension of P.

Linearizability of ω

Proof.

For the reversal, prove FUF.

- Let $\{X_i\}_{i<n}$ be a finite family of finite sets.
- Set the partial order $P = \bigoplus_{i<n}(X_i + \{a_i\})$.



- P is ω-like. Apply the HP and obtain an ω-like linear extension L. Let a_j be the L-maximum of the a_i's.
- Then $\bigcup_{i<n} X_i$ is finite because it is included in the set of the L-predecessors of a_j. □

Linearizability of $\omega + \omega^*$

Theorem

"$\omega + \omega^*$ is linearizable" is equivalent to ACA_0 *over* RCA_0.

Proof.

In ACA_0 we can sort out the elements with finitely many predecessors from the others and apply $\mathrm{B}\mathbf{\Sigma}^0_2$.

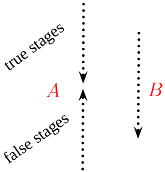
For the reversal, we reason in RCA_0 and prove that for every one-to-one function $f\colon \mathbb{N} \to \mathbb{N}$, the range of f exists.

- Let $A = \{a_n \colon n \in \mathbb{N}\}$ be the $\omega + \omega^*$ linear order given by the false and the true stages of f.
- $n \in \mathbb{N}$ is *false* if $(\exists m > n)(f(m) < n)$.

Linearizability of $\omega + \omega^*$

Proof.

- We define an $\omega + \omega^*$-like partial order P putting together A with a linear order B of order type ω^*.



- By HP, there is an $\omega + \omega^*$-like linear extension L.
- "n is false" is $\mathbf{\Pi}^0_1$ definable by $(\forall m)(a_n \leq_L b_m)$. □

Embeddability of ω

Theorem

"ω is embeddable" is equivalent to ACA_0 *over* RCA_0.

Proof.

Assume ACA_0.

- Let P an ω-like partial order.
- By $\mathrm{B}\mathbf{\Sigma}^0_2$, there exists an ω-like linear extension L.
- Define an embedding $h\colon P \to \omega$ by letting

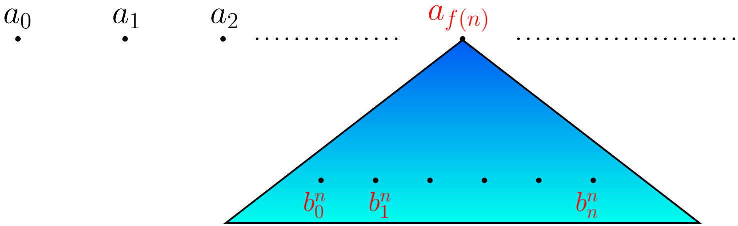
$$h(x) = |\{y \in P\colon y <_L x\}|.$$

Embeddability of ω

Proof.

For the reversal, assume that ω is embeddable.

- Let $f\colon \mathbb{N} \to \mathbb{N}$ be a one-to-one function. We prove that the range of f exists.
- Define an ω-like partial order P as in the picture.



- Apply the HP and obtain an embedding $h\colon P \to \omega$.
- Check that $(\exists n)(f(n) = m)$ iff $(\exists n < h(a_m))(f(n) = m)$. □

References

P. Cholak and C. Jockusch and T. Slaman. *On the strength of Ramsey's theorem for pairs*, J. Symbolic Logic, 66(1):1–55, 2001.

R. Fraïssé. *Theory of relations*, volume 145 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, revised edition, 2000.

E. Frittaion and A. Marcone. *Linear extensions of partial orders and reverse mathematics* (to appear in MLQ).

D. Hirschfeldt and R. Shore. *Combinatorial principles weaker than Ramsey's theorem for pairs*, J. Symbolic Logic, 72(1):171–206, 2007.

J. Hirst, *Combinatorics in Subsystems of Second Order Arithmetic*, The Pennsylvania State University, PhD thesis, 1987.

Stephen G. Simpson. *Subsystems of second order arithmetic*, *Perspectives in Logic*, Cambridge University Press, Cambridge, 2009.